5. Theory of Angular Distributions

5.1. Helicity Formalism

Typically when examining collisions, a fixed direction is chosen in order to classify the various polarization states of a particle. In the helicity formalism, states are labelled by the component of the total angular momentum along the direction of the particle motion. This avoids the relativistic complications inherent in breaking up the angular momentum operator into a spin- and an orbital- part[67].

Another advantage of the helicity formalism pertains to two-particle helicity states. Within the center-of-mass frame, the orbital angular momentum is always perpendicular to the relative motion of the two particles. Defining the helicity of the state as

$$\lambda = \frac{\vec{J} \bullet \vec{p}}{|\vec{p}|}, \tag{5.1}$$

the component of spin along the direction of relative motion will also be the component of the total angular momentum along this direction[68], since $\vec{J} = \vec{L} + \vec{S}$ and $\vec{L} \bullet \vec{p} = 0$.

The helicity quantum number λ is also invariant with respect to ordinary rotations. In other words performing a rotation on a state will change the direction of the momentum (keeping its magnitude constant), but the helicity quantum number does not change during this operation. However, as seen by the equation above, if the momentum vector is rotated by 180 degrees, the helicity quantum number changes sign.

Given a set of total angular momentum operators (J_x, J_y, J_z) and three Euler angles (α, β, γ) it is customary to describe a rotation by

(5.2) $$R(\alpha, \beta, \gamma) = e^{-i\alpha J_x} \; e^{-i\beta J_y} \; e^{-i\gamma J_z}.$$

In the helicity formalism, one wants to rotate the momentum vector away from the z-axis to the direction (θ,φ) in a unique manner, so (α, β, γ) is (φ, θ, $-\varphi$).

In the basis formed by angular momentum eigenvectors, the matrix element of this rotation matrix is

(5.3) $$\langle J' M' | \; R(\alpha, \beta, \gamma) \; | J M \rangle = \delta_{JJ'} \, D^J_{M M'} \, (\alpha, \beta, \gamma).$$

The rotation matrix D for a specific total spin J can thus be found by inserting the expression for a general Euler rotation between two states of the same J. This rotation matrix is represented as

$$D^{J}_{M M'}{}^{*}(\alpha, \beta, \gamma) = e^{-i\alpha M}\, d^{J}_{M M'}(\beta)\, e^{-i\gamma M'}\,, \tag{5.4}$$

where

$$d^{J}_{M M'}(\beta) = \langle\, J\, M' \mid e^{-i\beta J_y} \mid J\, M \rangle. \tag{5.5}$$

In other words, when we perform a rotation upon a particular angular momentum state, the rotation matrix supplies the coefficients of the new rotated state in terms of all the 2J+1 angular momentum states making up the basis for this vector space[69].

Using the above tools and a knowledge of the S-matrix, one may determine the matrix element for a process $a + b \rightarrow c + d$ in the center-of-mass, where a and b lie along the Z-axis, and the relative momentum of c and d is rotated to the direction (θ,φ) for one given set of helicities.

Suppose that initially we have two particles a and b heading toward each other along the z-axis, and the collision produces two other particles c and d. In our case, a and b might represent the proton and the antiproton. The relative momentum vector of c and d is rotated by the angles θ and φ away from the z-axis in the center-of-mass frame. Jacob and Wick express this S-matrix element as

(5.6) $$C = \langle \theta \phi \lambda_c \lambda_d | S(E) | 0 0 \lambda_a \lambda_b \rangle .$$

One can then insert a complete set of states before and after the S(E):

(5.7) $$C = \sum_{J,M} \sum_{J',M'} \{ \langle \theta\phi\lambda_c\lambda_d | JM\lambda_c\lambda_d \rangle \langle JM\lambda_c\lambda_d | S(E) | J'M'\lambda_a\lambda_b \rangle \langle J'M'\lambda_a\lambda_b | 00\lambda_a\lambda_b \rangle \} .$$

This may be simplified by the applying

(5.8) $$\langle E J M \lambda_c \lambda_d | S | E' J' M' \lambda_a \lambda_b \rangle = \delta(E - E')\, \delta_{JJ'}\, \delta_{MM'} \langle \lambda_c \lambda_d | S^J(E) | \lambda_a \lambda_b \rangle$$

and the definition of the rotation matrix

(5.9) $$\langle \theta \phi \lambda_1 \lambda_2 | J M \lambda_1 \lambda_2 \rangle = D^J_{M \lambda}(\phi, \theta, -\phi),$$

where λ is now the difference of the two helicity quantum numbers, $\lambda_1 - \lambda_2$.

The end result for the matrix element in the helicity formalism may be expressed as

$$\text{(5.10)} \qquad \frac{1}{4\,p}\sum_{J}\left(2J+1\right)\left\langle \lambda_c\lambda_d\right| S^J(E)\left|\lambda_a\lambda_b\right\rangle D^{J\,*}_{\lambda\mu}\left(\phi,\theta,-\phi\right).$$

Here λ is the difference of the final state helicities (c and d), and μ is the difference of the initial state helicities (a and b). The total amplitude considers all possible values of total angular momentum J. However, we only concern ourselves with J=1 because of the vector nature of the Ψ and Ψ'.

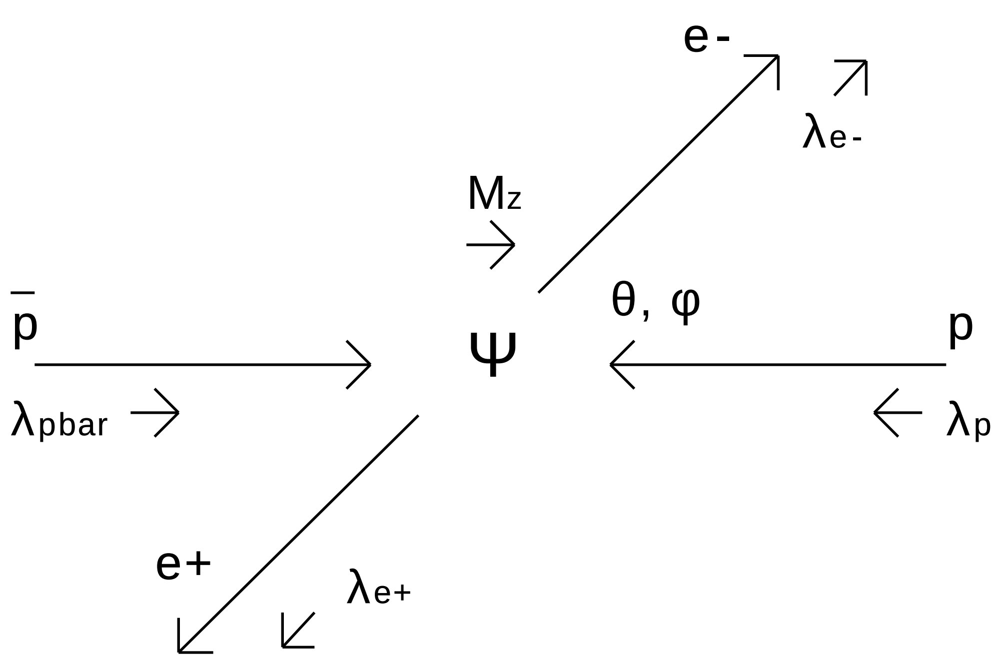

Figure 5.1 : Decay of the Ψ(Ψ') in the helicity formalism.

For the decay process $\bar{p}\ p \Rightarrow \Psi \Rightarrow e^+ e^-$ shown in Figure 5.1, we must go through two stages: $\bar{p}\,p \rightarrow \Psi$, and then $\Psi \rightarrow e^+e^-$. To derive the matrix element , one first multiplies an amplitude by a rotation matrix for both the production and decay stages, and then sums over all possible intermediate states M_z of the vector state Ψ, since the Ψ is unobserved[70]:

$$(5.11)\quad M\left(\lambda_{\bar{p}p}, \lambda_{e^+e^-}, J\right) = \sum_{M_z} \left\{ C_{M_z} D^{J*}_{\lambda_{\bar{p}p}, M_z}(0,0,0) \right\} \left\{ B_{\lambda_{e^+e^-}} D^{J*}_{M_z, \lambda_{e^+e^-}}(\phi, \theta, -\phi) \right\}.$$

Here $\lambda_{\bar{p}p}$ refers to the difference in helicity between the proton and antiproton, and $\lambda_{e^+e^-}$ refers to the difference in helicity between the electron and the positron. The first rotation matrix above is a delta function, meaning $\lambda_{\bar{p}p}$ = M_z . The first term in brackets describes the production of a charmonium state at the origin with a polarization given by M_z, and so only one term contributes from this sum. Hence we have

$$(5.12)\qquad M\left(\lambda_{\bar{p}p}, \lambda_{e^+e^-}, J\right) \propto \left\{ C_{\lambda_{\bar{p}p}} \right\} \left\{ B_{\lambda_{e^+e^-}} D^{J}_{\lambda_{\bar{p}p}, \lambda_{e^+e^-}}(\phi, \theta, -\phi) \right\}.$$

After this matrix element is squared, it must be averaged over the initial $\bar{p}p$ states and summed over the final e^+e^- states to get the differential cross section. This means summing over all possible helicity combinations of the proton and antiproton, and then summing over all possible combinations for the electron and positron. There are four such helicity states for the proton and antiproton $(\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow)$, which means that there are two possible ways to produce helicity zero. The electron and positron state cannot have a helicity zero term if we neglect the masses of the electrons.

Since the helicity of each individual proton or electron is +1/2 or -1/2, there exist five different helicity amplitudes in our exclusive decays: C_{-1}, C_0, C_1, B_1, and B_{-1}. Although the C's and B's are independent of each other, the following parity relations apply:

(5.13) ;

$$B_{\lambda_{e^+e^-}} = \eta_{\psi}\eta_{e^+e^-}(-1)^{S_\psi - (S_{e^+} + S_{e^-})} B_{-\lambda_{e^+e^-}}$$
$$= (-1)(-1)(-1)^{1-\left(\frac{1}{2}+\frac{1}{2}\right)} B_{-\lambda_{e^+e^-}}$$

.

$$C_{\lambda_{\bar{p}p}} = \eta_{\bar{p}p}\eta_{\psi}(-1)^{(S_{\bar{p}} + S_p) - S_\psi} C_{-\lambda_{\bar{p}p}}$$
$$= (-1)(-1)(-1)^{\left(\frac{1}{2}+\frac{1}{2}\right)-1} C_{-\lambda_{\bar{p}p}}$$

As a result, $B_1 = B_{-1}$ and $C_1 = C_{-1}$. We now square the matrix element above and sum over all possible helicity combinations for the proton and antiproton, and then sum over all possible combinations for the electron and positron:

$$\langle M^2 \rangle \propto \sum_{\lambda_{\bar{p}p},\, \lambda_{e^+e^-}} B^2_{\lambda_{e^+e^-}}\, C^2_{\lambda_{\bar{p}p}} \left(d^1_{\lambda_{\bar{p}p},\, \lambda_{e^+e^-}}(\theta) \right)^2. \tag{5.14}$$

In the process of squaring the rotation matrix, the exponential factors have canceled with their complex conjugates, so the angular distribution does not depend upon the azimuthal angle of the electrons in the center of mass frame. The sum is now performed over $\lambda_{\bar{p}\,p}$ = -1, 0, 0, 1 and $\lambda_{e^+e^-}$ = -1, 1. Since we are only interested in the shape of the angular distribution, and $B_1^{\,2}$ is common to every term in this sum due to parity, we may safely factor it out.

Using the following *d* functions for J=1,

$$d^1_{11}(\theta) = d^1_{-1-1}(\theta) = \cos^2\left(\frac{\theta}{2}\right); \tag{5.15}$$

$$d^1_{1-1}(\theta) = d^1_{-11}(\theta) = \sin^2\left(\frac{\theta}{2}\right);$$

and

$$d^1_{01}(\theta) = d^1_{0-1}(\theta) = \sqrt{2}\,\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right);$$

one arrives at the final form of the angular distribution of the electrons in the center-of- mass for this exclusive process:

$$\langle M^2 \rangle \propto 1 + \alpha \cos^2(\theta) , \tag{5.16}$$

where

$$\alpha = \frac{C_1^2 - 2\, C_0^2}{C_1^2 + 2\, C_0^2} .$$

Since we have taken the electrons to be massless, the angular distribution parameter for our exclusive processes is solely dependent upon the manner in which the J=1 charmonium state was produced. If we had produced charmonium by colliding electron beams instead of annihilating protons and antiprotons, the sum over the initial e^+e^- helicity states would have excluded any production of helicity zero charmonium, and α would become unity. This has already been confirmed by previous e^+e^- experiments for both the J/Ψ and Ψ'[71,72]. This parameter is less than one for the case of antiproton-proton annihilation in general because the mass of the proton is much greater than the mass of the electron.

5.2. The angular distribution parameter

In exclusive decays, one cannot distinguish whether the electrons came from a charmonium decay, or directly from the continuum channel $\bar{p}\,p \rightarrow e^+e^-$. Thus the parameter α has a strong contribution resulting from the 3-gluon exchange in the production of charmonium (i.e. the lowest order Feynman graph shown in Figure 5.2), and an electromagnetic contribution resulting from the annihilation of the proton and antiproton into a virtual photon, which may or may not include an intermediate vector charmonium state. Furthermore, another electromagnetic contribution stems from the replacement of one of those 3 gluons by a photon.

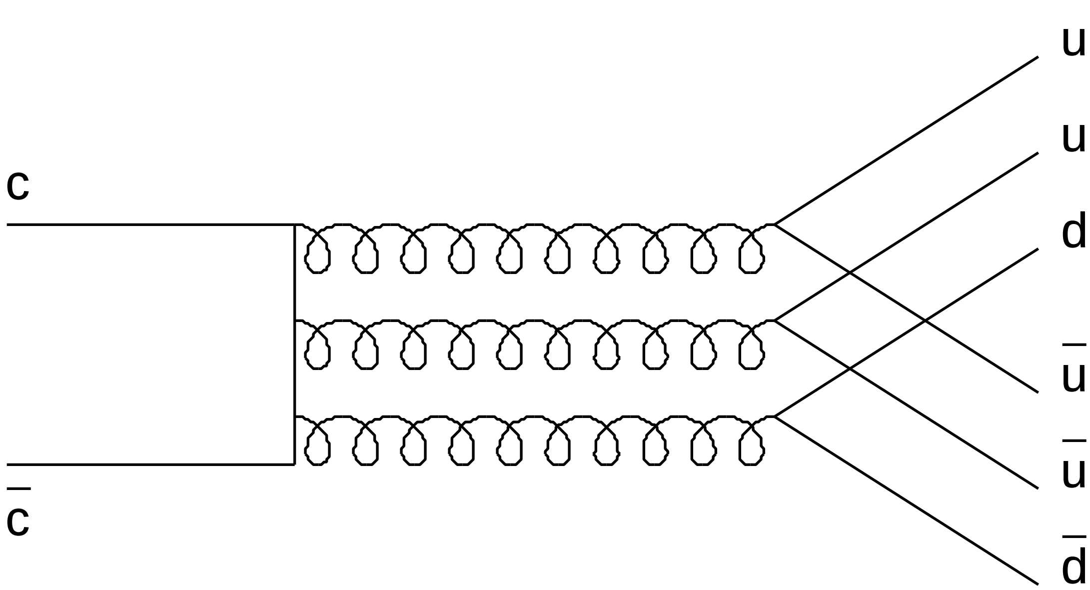

Figure 5.2 : Lowest order Feynman graph for the strong contribution to the angular distribution parameter.

However, measurements of the time-like electromagnetic form factors away from the charmonium region reveal that the cross-section for the continuum channel is negligible at the J/Ψ and Ψ', on the order of a picobarn[73]. Futhermore, calculations show that the electromagnetic contributions to the angular distribution parameter are on the order of 5%.

One may also question whether these exclusive decays also receive a contribution from the $c\bar{c}g$ Fock state (charm quark + anti-charm quark + glue) in addition to the diagram in Figure 5.2. In general, the Fock state representation includes all quantum fluctuations of the hadron wavefunction[74]. In the case of P-wave states, the valence $c\bar{c}$ state is suppressed in exclusive decays and the $c\bar{c}g$ Fock state contributes. Fortunately in S-wave exclusive decays like ours the situation is reversed. Higher-order Fock states like $c\bar{c}g$ are suppressed by factors of the charmed quark mass and velocity[75].

Usually when one discusses the angular distribution of the continuum channel, one talks about extracting a pair of electromagnetic form factors of the proton. Due to the vector-like nature of the intermediate photon, Lorentz invariance allows the proton-antiproton-photon vertex to be written in the following manner:

$$(5.17) \qquad \Gamma^{\mu} = e\bar{v}(p')\left[\gamma^{\mu}F_1\left(Q^2\right) - \frac{(p+p')^{\mu}}{2M}F_2\left(Q^2\right)\right]u(p) \ ,$$

where F_1 and F_2 refer to the helicity-conserving and helicity-flip parts of this vertex and are referred to as Dirac form factors. From this one can derive the form of the angular distribution for this process[76]

$$(5.18) \quad \frac{d\sigma}{d(\cos\theta^*)} = \frac{\pi\alpha^2}{8EP}\left[\left|G_M\right|^2\left(1+\cos^2\theta^*\right) + \frac{4m_p^{\,2}}{s}\left|G_E\right|^2 \sin^2\theta^*\right],$$

where E and P are the center-of-mass energy and momentum of the antiproton, θ^* is the polar decay angle of the antiproton in the center-of-mass frame, and the Sachs electromagnetic form factors are defined as

$$(5.19) \qquad G_M = F_1 + F_2.$$

and

$$G_E = F_1 + \frac{q^2}{4m_p^{\,2}}F_2.$$

As a result, the angular distribution parameter may be written as

$$(5.20) \qquad \alpha_P = \frac{E_{CM}^{\;2} - 4\left|\frac{G_E}{G_M}\right|^2 m_p^{\,2}}{E_{CM}^{\;2} + 4\left|\frac{G_E}{G_M}\right|^2 m_p^{\,2}}.$$

In the case of charmonium , one can also extract form factors like these from the angular distribution of charmonium decays since both the virtual photon and the spin 1 charmonium state are vector states. However, one cannot call these form factors electromagnetic . These “strong” form factors can also be evaluated by applying Lorentz invariance at the vertex.

The distinction can be clarified by the derivation of the form factor F_1 by Brodsky and Lepage[77]. They break up the electromagnetic form factor into 3 parts: a) an amplitude φ for finding the three-quark valence state in the incoming proton, b) an amplitude T_H for this state to scatter with the photon producing 3 quarks with roughly collinear momenta, and c) an amplitude φ* for this final quark state to form into a hadron:

$$F_1(Q^2) = \int_0^1 [dx] \int_0^1 [dy]\, \varphi^*\left(y_i, \tilde{Q}_y\right) T_H(x_i, y_i, Q)\, \varphi\left(x_i, \tilde{Q}_x\right), \tag{5.21}$$

where

$$[dx] = dx_1\, dx_2\, dx_3 \delta(1 - x_1 - x_2 - x_3) \tag{5.22}$$

and
$$\tilde{Q}_x = \min(x_i Q) ,$$

and x represents the fraction of the longitudinal momentum taken up by each valence quark.

The formation of charmonium from proton-antiproton annihilation really involves two distinctly different processes[78]: An interaction between the various quarks and gluons that is primarily perturbative in nature, and a recombination of the quarks into a proton (or breaking up thereof) that is basically non-perturbative.

T_H , the "hard scattering amplitude", includes diagrams like those in Figure 5.1 with the exchange of gluons. In the electromagnetic case such diagrams refer only to the exchange of the virtual photon. So for any process, T_H addresses the hard scattering among the gluons and three quarks of the hadron, which can be treated perturbatively.

The "quark distribution amplitude" φ, a function of Q^2, involves soft processes that recombine the produced quarks into hadrons. At charmonium energies, such soft processes are not treatable by perturbative methods in QCD. The distribution amplitude can be thought of as a probability of finding a valence quark in the proton carrying a fraction X_i of the proton's momentum and carrying a transverse momentum less than some scale Q. As a result, these valence quarks may be treated as moving collinearly up to this scale Q[79]. Furthermore, these amplitudes are independent of the exclusive decay and are related to the total hadronic wave function.

The soft processes responsible for the recombination phase are typically studied via QCD sum rules or lattice calculations. QCD sum rules allow one to calculate different moments of the distribution amplitudes. For instance the (n1, n2, n3)-th moment of the proton's distribution amplitude is

$$\int_{x_1,x_2,x_3} \varphi\left(x_1, x_2, x_3, Q^2\right) x_1^{n_1} x_2^{n_2} x_3^{n_3} dx_1 dx_2 dx_3. \quad (5.23)$$

Perturbative QCD yields a general expression for the distribution amplitude at a very high energy that is an infinite sum of terms with an unknown non-perturbative parameter which is related to these (n1, n2, n3)-th moments. The different theoretical models for the distribution amplitudes all evaluate as many moments as possible at a given energy, truncate the general perturbative expression, and find the unknown coefficients using QCD sum rules[80]. However, it is not completely determined at this time whether one can apply the general QCD expression or apply such a truncation to N terms at lower energies.

5.3 Comparison of Theoretical Predictions

Brodsky and Lepage[81] predicted that if one neglects the mass of the constituent quarks, then the total hadronic helicity of all the constituent valence quarks is conserved. Furthermore, the angular distribution of Ψ decays into $\bar{p}$ p must have the same form, 1 + $\cos^2(\theta)$, that the direct decay of e^+e^- into $\bar{p}$ p via a photon has because the J/Ψ is a vector particle like the virtual photon.

One can still favor a spin-one gluon and have an α that is less than one if one considers various mass effects on the decay. Claudson, Glashow, and Wise took into account the effect of the total baryon mass as a whole and neglected the contribution of the form factor F_2[82],

(5.24) $$\frac{d\Gamma(J/\Psi \to B\bar{B})}{d(\cos\theta)} \propto 1 + \frac{M_\Psi^2 - 4m_B^2}{M_\Psi^2 + 4m_B^2}\cos^2\theta,$$

which predicts α = 0.463 for the J/Ψ and α = 0.589 for the Ψ'. Effectively, this derivation neglects the flipping of the constituent quarks' spins in the proton.

Carimalo considered mass effects on the quark level by (a) assigning each constituent quark to have an effective mass of 1/3 the mass of the proton and (b) assuming a nonrelativistic static quark model , in which all quarks share the momentum equally[83]. He derives the angular distribution coefficient as

(5.25) $$\alpha = \frac{(1+u)^2 - u(1+6u)^2}{(1+u)^2 + u(1+6u)^2},$$

where $$u = \frac{m_p^2}{M_\psi^2}.$$

Disregarding the small electromagnetic correction, this results in α = .688 for the J/Ψ and α =.802 for the Ψ'.

At very high energies, the angular distribution parameter should be one, regardless of what kind of exclusive decay occurs, since the energy of the resonance will be much larger than the mass of the proton. At these energies, perturbative QCD reigns. At lower energies, distances between the valence quarks of the proton may become larger. Hence the confinement QCD potential, which is not well modelled perturbatively, comes into play. Furthermore, the masses of the quarks and hadrons, and perhaps an effective mass for the gluon, have an impact as well. Since the mass of the Ψ is lower than the mass of the Ψ', the angular distribution parameter α at the Ψ should be lower than α at the Ψ' for our exclusive decays into e^+e^-.

The world experimental average[84] of the angular distribution parameter for the Ψ prior to 1989 was 0.63 ±.08. The derivation of Claudson, Glashow, and Wise was able to demonstrate that corrections due to mass effects can be substantial, but it also shows that one cannot neglect the impact of the individual quarks. So of the first three predictions presented so far, a non-relativistic treatment, whereby each of the valence quarks of the proton share the momentum equally, appears to predict the angular distribution parameter at the J/ Ψ very well.

More involved calculations of the angular distribution parameter call upon different models for the quark distribution amplitudes, different applications of QCD sum rules discussed in Section 5.2, and various forms of the coupling constant α_s used in the calculation. The predictions shown in Tables 5.1-5.2 reveal that the width from charmonium to $\bar{p}p$ is also significant. The non-relativistic treatment of Carimalo[83] does not hold up as well under this additional constraint, and it appears that the "heterotic" solution of Stefanis and Bergmann [85] matches both (a) the world average for the angular distribution parameter and (b) the Particle Data Group value for the width $\Gamma(J/\Psi \rightarrow \bar{p}p)$ of 188 eV.

Also presented in Table 5.2 are the respective theoretical expectations[86] for the angular distribution parameter at the Ψ' and the respective decay widths to $\bar{p}p$. The same models which predict this parameter and the $\bar{p}p$ decay width[87] at the Ψ are used to predict it at the Ψ'. According to the Particle Data Group the value of the decay width to $\bar{p}p$ is 53 eV at the Ψ'.

DA Model	α_p	$\Gamma(J/\Psi \rightarrow \bar{p}p)$ (eV)
Brodsky and Lepage[81]	1	-
Claudson, Glashow, and Wise[82]	0.46	-
Carimalo(Non-relativistic)[83]	0.688	0.2
Asymptotic[78]	0.667	2.6
Chernyak and Zhitnitsky[88]	0.561	58.7
Chernyak, Oglobin, and Zhitnitsky[89]	0.565	82.6
King and Sachrajda[90]	0.591	125.5
Gari and Stefanis[91]	0.963	16.8
Stefanis and Bergmann (heterotic)[85]	0.689	167.1

Table 5.1 : Theoretical Predictions of the angular distribution parameter and partial width to $\bar{p}p$ at the J/Ψ.

DA Model	α_p	$\Gamma(\Psi' \to \bar{p} p)$ (eV)
Brodsky and Lepage[81]	1	-
Claudson, Glashow, and Wise[82]	0.59	-
Carimalo(Non-relativistic)[83]	0.802	0.02
Asymptotic[78]	0.782	0.21
Chernyak and Zhitnitsky[88]	0.683	5.47
Chernyak, Oglobin, and Zhitnitsky[89]	0.687	6.66
King and Sachrajda[90]	0.712	10.31
Gari and Stefanis[91]	0.986	1.07
Stefanis and Bergmann[85] (heterotic)	0.790	14.00

Table 5.2 : Theoretical Predictions of the angular distribution parameter and $\bar{p} p$ at the Ψ' .